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Radial focusing of a relativistic electron beam in a bipotential electrostatic lens

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The focusing of a relativistic electron beam in a bipotential electrostatic lens is discussed. An iterative scheme for the solution of the paraxial ray equation is used to derive approximate analytic formulas for the lens parameters and lens transfer matrix elements. The formulas are compared to results of direct numerical integration of the paraxial ray equation.

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I. INTRODUCTION

Chen and Reiser [1] recently presented a discussion of the focusing properties of a bipotential electrostatic lens in the thin lens approximation, extending the existing paraxial theory [2] to the relativistic regime. Results of such an analysis are applicable to electrostatic acceleration systems as well as linear induction accelerators for which the gap crossing time of a particle is short compared with the time scale on which the electric field varies. The present communication is an extension of the work of Chen and Reiser in which we develop an iterative scheme for the approximate solution of the relativistically correct paraxial ray equation. Analytic formulas are derived for the focal lengths f_1, f_2 and the distances of the principal planes from the center of the lens, d_1, d_2. The approximate analytic formulas of Ref. [1] can be interpreted as the first step ($n=1$) of the iterative scheme developed below. (Some discrepancies between our numerical results and those of Ref. [1] will be noted.) We carry the process one step further ($n=2$), and the formulas so derived are found to be in excellent agreement with numerical integration of the paraxial ray equation over a wide range of initial and final beam voltages.

The paper is organized as follows. Section II contains a description of the bipotential lens model and a statement of the paraxial ray equation to be solved. An iterative scheme for solving the equation is derived in Sec. III. Analytic formulas for the lens parameters are presented in Sec. IV and are compared to results from direct numerical integration.

II. LENS MODEL AND RELATIVISTIC PARAXIAL RAY EQUATION

We consider a symmetrical two-cylinder lens with radius b and negligibly small separation $\delta \ll b$ (see Fig. 1). For this case, the electric field distribution along the axis of the pipe is described to a good degree of accuracy by the potential function [1,2]

$$V(z)=\frac{(V_1+V_2)}{2}+\frac{(V_2-V_1)}{2}\tanh(\alpha z)\ , \tag{1}$$

where $\alpha=1.32/b$ and $z=0$ at the center of the gap. (We have followed the notation of Ref. [1] wherever possible.) The relativistic factor γ then varies according to

$$\gamma(z)=\frac{(\gamma_1+\gamma_2)}{2}+\frac{(\gamma_2-\gamma_1)}{2}\tanh(\alpha z)\ . \tag{2}$$

The relativistically correct paraxial ray equation can be written as [3]

$$r''+\frac{\gamma'}{\beta^2\gamma}r'+\frac{\gamma''}{2\beta^2\gamma}r=0\ , \tag{3}$$

where the prime in Eq. (3) denotes d/dz, and γ, γ', and γ'' are determined from Eq. (2). Introducing the "reduced" variable

$$R(z)\equiv r(z)[\gamma^2(z)-1]^{1/4} \tag{4}$$

results in the equation

$$R''(z)+G(z)R(z)=0\ , \tag{5}$$

where

$$G(\xi)=\frac{A\alpha^2}{4}\left[\frac{3(1-\xi^2)^2}{D^2}+\frac{(1-\xi^2)^2}{D}\right]\ . \tag{6}$$

$\xi\equiv\tanh(\alpha z)$, and $D\equiv A\xi^2+B\xi+C$, where

$$A\equiv\frac{(\gamma_2-\gamma_1)^2}{4}=\frac{\Delta\gamma^2}{4}\ , \tag{7}$$

$$B\equiv\frac{\Delta\gamma}{2}(\gamma_1+\gamma_2)\ , \tag{8}$$

$$C\equiv\frac{(\gamma_1+\gamma_2)^2}{4}-1\ . \tag{9}$$

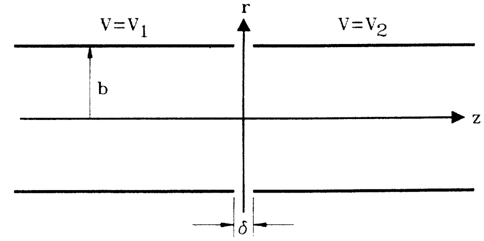

FIG. 1. Bipotential electrostatic lens of radius b and gap separation δ.

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In what follows, we will also make use of the functional definitions

$$\bar{G}(\xi)\equiv\frac{G(\xi)}{\alpha^2(1-\xi^2)}=\frac{A}{4}\left[\frac{3(1-\xi^2)}{D^2}+\frac{(1-\xi^2)}{D}\right] \quad (10)$$

and

$$\bar{F}(\xi)\equiv\int_{-1}^{\xi}\bar{G}(\xi_1)d\xi_1 \ . \quad (11)$$

III. ITERATIVE SOLUTION OF THE PARAXIAL RAY EQUATION

Consider the trajectory $r(z)$ shown schematically in Fig. 2. The focal length f_2 on the downstream (image) side of the lens is given by

$$\begin{aligned}\frac{1}{f_2}&=-\lim_{z\to\infty}\frac{r'(z)}{r_1}=-\frac{(\gamma_1^2-1)^{1/4}}{(\gamma_2^2-1)^{1/4}}\frac{1}{R_1}\lim_{z\to\infty}R'(z)\\&=-\frac{(\gamma_1^2-1)^{1/4}}{(\gamma_2^2-1)^{1/4}}\frac{1}{R_1}\lim_{z\to\infty}\int_{-\infty}^{z}R''(z_1)dz_1\\&=\frac{(\gamma_1^2-1)^{1/4}}{(\gamma_2^2-1)^{1/4}}\frac{1}{R_1}\int_{-\infty}^{\infty}G(z)R(z)dz \ ,\end{aligned} \quad (12)$$

where we have used Eq. (5). We can also see that

$$\begin{aligned}\Delta&=\lim_{z\to\infty}[r_1-r(z)+zr'(z)]\\&=r_1-\frac{1}{(\gamma_2^2-1)^{1/4}}\lim_{z\to\infty}[R(z)-zR'(z)] \ .\end{aligned} \quad (13)$$

Using the relation

$$R(z)-zR'(z)=R_1+\int_{-\infty}^{z}z_1G(z_1)R(z_1)dz_1 \ , \quad (14)$$

we can write

$$\frac{\Delta}{r_1}=\left[1-\frac{(\gamma_1^2-1)^{1/4}}{(\gamma_2^2-1)^{1/4}}\right]-\frac{(\gamma_1^2-1)^{1/4}}{(\gamma_2^2-1)^{1/4}}\int_{-\infty}^{\infty}zG(z)R(z)dz \ . \quad (15)$$

Making use of Eqs. (12)–(15), we define the following iterative scheme:

$$R^{(0)}(z)=R_1=r_1(\gamma_1^2-1)^{1/4}=\text{const} \ , \quad (16)$$

$$\frac{1}{f_2^{(n)}}=\frac{(\gamma_1^2-1)^{1/4}}{(\gamma_2^2-1)^{1/4}}\frac{1}{R_1}\int_{-\infty}^{\infty}G(z)R^{(n-1)}(z)dz \ , \quad (17)$$

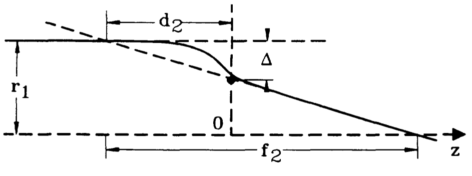

FIG. 2. Schematic of trajectory $r(z)$ with $r(-\infty)=r_1, r'(-\infty)=0$. Focal length f_2 and distance to the principal plane d_2 shown.

$$\left[\frac{\Delta}{r_1}\right]^{(n)}=\left[1-\frac{(\gamma_1^2-1)^{1/4}}{(\gamma_2^2-1)^{1/4}}\right]-\frac{(\gamma_1^2-1)^{1/4}}{(\gamma_2^2-1)^{1/4}}\frac{1}{R_1}\int_{-\infty}^{\infty}zG(z)R^{(n-1)}(z)dz \ , \quad (18)$$

$$d_2^{(n)}=f_2^{(n)}\left[\frac{\Delta}{r_1}\right]^{(n)} \ , \quad (19)$$

$$(R'(z))^{(n)}=-\int_{-\infty}^{z}G(z_1)R^{(n-1)}(z_1)dz_1 \ , \quad (20)$$

$$R^{(n)}(z)=R_1+\int_{-\infty}^{z}(R'(z_1))^{(n)}dz_1 \ , \quad (21)$$

with $n=1,2,\ldots$. Equations (16)–(21) can be used to determine the focal length f_2 and the distance d_2 of the corresponding principal plane from the center of the lens. The focal length f_1 on the upstream (object) side of the lens and the distance d_1 may be found by interchanging γ_1 and γ_2 in the expressions for f_2 and d_2, respectively. Explicit analytic formulas for $n=1$ and 2 have been worked out and are given below.

IV. ANALYTIC AND NUMERICAL RESULTS

A. First order ($n=1$) expressions

$$\frac{1}{f_2^{(1)}}=\frac{(\gamma_1^2-1)^{1/4}\alpha}{(\gamma_2^2-1)^{1/4}}\int_{-1}^{1}\bar{G}(\xi)d\xi \ , \quad (22)$$

$$\left[\frac{\Delta}{r_1}\right]^{(1)}=\left[1-\frac{(\gamma_1^2-1)^{1/4}}{(\gamma_2^2-1)^{1/4}}\right]-\frac{(\gamma_1^2-1)^{1/4}}{(\gamma_2^2-1)^{1/4}}\int_{-1}^{1}\bar{G}(\xi)\tfrac{1}{2}\ln\frac{(1+\xi)}{(1-\xi)}d\xi \ . \quad (23)$$

In terms of the following integrals

$$S_1(\gamma_1,\gamma_2)=\int_{-1}^{1}\bar{G}(\xi)d\xi \ , \quad (24)$$

$$S_2(\gamma_1,\gamma_2)=\int_{-1}^{1}\bar{G}(\xi)\tfrac{1}{2}\ln\frac{(1+\xi)}{(1-\xi)}d\xi \ , \quad (25)$$

the $n=1$ expressions for the focal lengths $f_{1,2}$ and $d_{1,2}$ are as follows:

$$\frac{1}{f_2}=\frac{(\gamma_1^2-1)^{1/4}\alpha}{(\gamma_2^2-1)^{1/4}}S_1(\gamma_1,\gamma_2) \ , \quad (26)$$

$$f_1=\frac{(\gamma_1^2-1)^{1/2}}{(\gamma_2^2-1)^{1/2}}f_2 \ , \quad (27)$$

$$\frac{d_2}{f_2}=\left[1-\frac{(\gamma_1^2-1)^{1/4}}{(\gamma_2^2-1)^{1/4}}\right]-\frac{(\gamma_1^2-1)^{1/4}}{(\gamma_2^2-1)^{1/4}}S_2(\gamma_1,\gamma_2) \ , \quad (28)$$

$$\frac{d_1}{f_1}=\left[1-\frac{(\gamma_2^2-1)^{1/4}}{(\gamma_1^2-1)^{1/4}}\right]-\frac{(\gamma_2^2-1)^{1/4}}{(\gamma_1^2-1)^{1/4}}S_2(\gamma_2,\gamma_1) \ . \quad (29)$$

Equation (26) is the same as Eq. (30) of Ref. [1], where

TABLE I. Results from direct numerical integration of Eq. (3) for the lens parameters $f_{1,2}$ and $d_{1,2}$ for various values of initial and final beam voltages.

V_1 (kV)	V_2 (kV)	b/f_2	d_2/b	b/f_1	$-d_1/b$
100	200	0.029 95	6.042 82	0.044 21	4.991 53
100	300	0.064 36	4.427 78	0.121 01	3.271 67
100	400	0.090 77	3.958 36	0.204 37	2.703 95
300	400	0.005 57	15.530 28	0.006 67	14.195 16
300	500	0.016 18	9.389 50	0.022 42	7.989 24
300	600	0.027 77	7.374 22	0.043 50	5.912 52
500	600	0.002 29	26.070 60	0.002 59	24.517 45
500	700	0.007 38	14.774 59	0.009 29	13.175 18
500	800	0.013 71	11.025 36	0.018 98	9.382 30

the integration in Eq. (24) was carried out. We have also carried out the integration in Eq. (25), and exact expressions for $S_1(\gamma_1,\gamma_2)$ and $S_2(\gamma_1,\gamma_2)$ are as follows:

$$S_1(\gamma_1,\gamma_2)=\frac{1}{4}\left\{\frac{[\gamma_1+\gamma_2+\frac{1}{2}(\gamma_1\gamma_2-5)]}{\Delta\gamma}\ln\left|\frac{\gamma_2^2-1}{\gamma_1^2-1}\right| -\frac{(\gamma_1\gamma_2-5)}{\Delta\gamma}\ln\left|\frac{\gamma_2+1}{\gamma_1+1}\right|-5\right\}, \quad (30)$$

$$S_2(\gamma_1,\gamma_2)=\frac{3}{16}\ln\left|\frac{\gamma_2^2-1}{\gamma_1^2-1}\right|+\frac{1}{32\Delta\gamma} \times\left\{[(\gamma_2-2)(\gamma_1-2)-9]\ln^2\left|\frac{\gamma_2+1}{\gamma_1+1}\right| -[(\gamma_2+2)(\gamma_1+2)-9]\ln^2\left|\frac{\gamma_2-1}{\gamma_1-1}\right|\right\}. \quad (31)$$

Note that $S_1(\gamma_1,\gamma_2)=S_1(\gamma_2,\gamma_1)$ and $S_2(\gamma_1,\gamma_2)=-S_2(\gamma_2,\gamma_1)$.

The lens matrix elements $[(a_{11},a_{12}),(a_{21},a_{22})]$ can be written in terms of $f_{1,2}$ and $d_{1,2}$ [see, for example, Eq. (35) of Ref. [1]]. Using direct numerical integration of Eq. (3), we have calculated tables for $f_{1,2},d_{1,2}$ (Table I), and for the transfer matrix elements (Table II). (We should point out that there are discrepancies between our results and those presented in Table I of Ref. [1]. We believe that the numbers in Tables I and II are correct to the accuracy shown.) The corresponding results obtained from the $n=1$ analytic expressions [Eqs. (26)–(31)] are given in Tables III and IV, respectively. In each case, comparison is made to the numerical results of Tables I and II.

TABLE II. Numerical results for the lens matrix elements.

V_1 (kV)	V_2 (kV)	a_{11}	a_{12}/b	$a_{21}b$	a_{22}
100	200	0.819 01	0.006 26	−0.029 95	0.827 06
100	300	0.715 01	0.015 81	−0.064 36	0.742 47
100	400	0.640 71	0.025 60	−0.090 77	0.689 57
300	400	0.913 48	0.001 05	−0.005 57	0.914 10
300	500	0.848 04	0.003 20	−0.016 18	0.851 21
300	600	0.795 21	0.005 80	−0.027 77	0.802 57
500	600	0.940 28	0.000 43	−0.002 29	0.940 44
500	700	0.890 91	0.001 41	−0.007 38	0.891 84
500	800	0.848 81	0.002 68	−0.013 71	0.851 22

TABLE III. (a) Results of $n=1$ analytical approximation for lens parameters $f_{1,2}$ and $d_{1,2}$ using Eqs. (26)–(31), and (b) fractional error in (a) relative to numerical results in Table I.

V_1 (kV)	V_2 (kV)	b/f_2	d_2/b	b/f_1	$-d_1/b$
			(a)		
100	200	0.030 22	5.987 01	0.044 60	4.951 92
100	300	0.065 81	4.326 40	0.123 73	3.213 02
100	400	0.094 03	3.816 40	0.211 72	2.633 15
300	400	0.005 58	15.507 02	0.006 68	14.174 99
300	500	0.016 26	9.344 88	0.022 52	7.954 65
300	600	0.028 01	7.309 56	0.043 87	5.866 84
500	600	0.002 29	26.055 15	0.002 59	24.503 35
500	700	0.007 40	14.744 47	0.009 31	13.149 80
500	800	0.013 77	10.981 20	0.019 05	9.347 57
			(b)		
100	200	0.008 94	−0.009 24	0.008 94	−0.007 94
100	300	0.022 50	−0.022 90	0.022 50	−0.017 93
100	400	0.035 99	−0.035 86	0.035 99	−0.026 18
300	400	0.001 47	−0.001 50	0.001 47	−0.001 42
300	500	0.004 63	−0.004 75	0.004 63	−0.004 33
300	600	0.008 55	−0.008 77	0.008 55	−0.007 73
500	600	0.000 59	−0.000 59	0.000 59	−0.000 58
500	700	0.002 00	−0.002 04	0.002 00	−0.001 93
500	800	0.003 91	−0.004 01	0.003 91	−0.003 70

TABLE IV. (a) Results of $n=1$ analytical approximation for lens matrix elements, and (b) fractional error in (a) relative to numerical results in Table II.

V_1 (kV)	V_2 (kV)	a_{11}	a_{12}/b	$a_{21}b$	a_{22}
			(a)		
100	200	0.819 07	0.000 55	−0.030 22	0.827 20
100	300	0.715 27	0.002 99	−0.065 81	0.743 35
100	400	0.641 13	0.006 81	−0.094 03	0.691 74
300	400	0.913 48	0.000 02	−0.005 58	0.914 10
300	500	0.848 06	0.000 16	−0.016 26	0.851 25
300	600	0.795 27	0.000 49	−0.028 01	0.802 69
500	600	0.940 28	0.000 00	−0.002 29	0.940 44
500	700	0.890 91	0.000 03	−0.007 40	0.891 85
500	800	0.848 83	0.000 11	−0.013 77	0.851 24
			(b)		
100	200	0.000 08	−0.912 73	0.008 94	0.000 17
100	300	0.000 36	−0.810 60	0.022 50	0.001 18
100	400	0.000 65	−0.733 98	0.035 99	0.003 15
300	400	0.000 00	−0.983 75	0.001 47	0.000 00
300	500	0.000 03	−0.951 28	0.004 63	0.000 04
300	600	0.000 08	−0.915 26	0.008 55	0.000 15
500	600	0.000 00	−0.993 66	0.000 59	0.000 00
500	700	0.000 01	−0.979 02	0.002 00	0.000 01
500	800	0.000 02	−0.960 41	0.003 91	0.000 03

B. Second order ($n=2$) expressions

$$\frac{1}{f_2^{(2)}}=\frac{(\gamma_1^2-1)^{1/4}\alpha}{(\gamma_2^2-1)^{1/4}} \times[S_1(\gamma_1,\gamma_2)+S_1(\gamma_1,\gamma_2)S_2(\gamma_1,\gamma_2) -S_3(\gamma_1,\gamma_2)]\ , \tag{32}$$

where

$$S_3(\gamma_1,\gamma_2)=\int_{-1}^{1}\bar{F}(\xi)\bar{G}(\xi)\ln\frac{(1+\xi)}{(1-\xi)}d\xi\ , \tag{33}$$

$$\left[\frac{\Delta}{r_1}\right]^{(2)}=\left[1-\frac{(\gamma_1^2-1)^{1/4}}{(\gamma_2^2-1)^{1/4}}\right] -\frac{(\gamma_1^2-1)^{1/4}}{(\gamma_2^2-1)^{1/4}}[S_2(\gamma_1,\gamma_2)+\tfrac{1}{2}S_2^2(\gamma_1,\gamma_2) -S_4(\gamma_1,\gamma_2)]\ , \tag{34}$$

where

$$S_4(\gamma_1,\gamma_2)=\int_{-1}^{1}\bar{F}(\xi)\bar{G}(\xi)\tfrac{1}{4}\ln^2\frac{(1+\xi)}{(1-\xi)}d\xi\ . \tag{35}$$

The integrals S_3 and S_4 were approximated by first doing an integration by parts and then approximating the resulting integrands with simple polynomial fits. Also, consider the factor in brackets in Eq. (32):

$$H(\gamma_1,\gamma_2)=[S_1(\gamma_1,\gamma_2) +S_1(\gamma_1,\gamma_2)S_2(\gamma_1,\gamma_2)-S_3(\gamma_1,\gamma_2)]\ .$$

It can be shown rigorously that $H(\gamma_1,\gamma_2)=H(\gamma_2,\gamma_1)$. With the approximation we used for the integral S_3, however, H was better approximated using $\gamma_1=\gamma_<$ and $\gamma_2=\gamma_>$ where $\gamma_<=\min(\gamma_1,\gamma_2)$ and $\gamma_>=\max(\gamma_1,\gamma_2)$. As a result, our $n=2$ expressions for $f_{1,2}$ and $d_{1,2}$ are

$$\frac{1}{f_2}=\frac{(\gamma_1^2-1)^{1/4}\alpha}{(\gamma_2^2-1)^{1/4}}[S_1(\gamma_1,\gamma_2)+S_1(\gamma_1,\gamma_2)S_2(\gamma_<,\gamma_>) -\tilde{S}_3(\gamma_<,\gamma_>)]\ , \tag{36}$$

$$f_1=\frac{(\gamma_1^2-1)^{1/2}}{(\gamma_2^2-1)^{1/2}}f_2\ , \tag{37}$$

$$\frac{d_2}{f_2}=\left[1-\frac{(\gamma_1^2-1)^{1/4}}{(\gamma_2^2-1)^{1/4}}\right]-\frac{(\gamma_1^2-1)^{1/4}}{(\gamma_2^2-1)^{1/4}} \times[S_2(\gamma_1,\gamma_2)+\tfrac{1}{2}S_2^2(\gamma_1,\gamma_2) -\tilde{S}_4(\gamma_1,\gamma_2)]\ , \tag{38}$$

$$\frac{d_1}{f_1}=\left[1-\frac{(\gamma_2^2-1)^{1/4}}{(\gamma_1^2-1)^{1/4}}\right]-\frac{(\gamma_2^2-1)^{1/4}}{(\gamma_1^2-1)^{1/4}} \times[S_2(\gamma_2,\gamma_1)+\tfrac{1}{2}S_2^2(\gamma_2,\gamma_1) -\tilde{S}_4(\gamma_2,\gamma_1)]\ . \tag{39}$$

In Eqs. (36)–(39), $\tilde{S}_3$ and $\tilde{S}_4$ denote the approximations to the integrals S_3 and S_4 defined below. Some ad-

TABLE V. (a) Results of $n=2$ analytical approximation for lens parameters $f_{1,2}$ and $d_{1,2}$ using Eqs. (36)–(45), and (b) fractional error in (a) relative to numerical results in Table I.

V_1 (kV)	V_2 (kV)	b/f_2	d_2/b	b/f_1	$-d_1/b$
			(a)		
100	200	0.029 96	6.042 10	0.044 21	4.991 00
100	300	0.064 37	4.427 53	0.121 02	3.270 99
100	400	0.090 71	3.960 89	0.204 23	2.702 72
300	400	0.005 57	15.529 96	0.006 67	14.194 90
300	500	0.016 19	9.388 87	0.022 42	7.988 76
300	600	0.027 77	7.373 37	0.043 51	5.911 91
500	600	0.002 29	26.070 40	0.002 59	24.517 26
500	700	0.007 38	14.774 17	0.009 29	13.174 84
500	800	0.013 71	11.024 74	0.018 98	9.381 83
			(b)		
100	200	0.000 13	−0.000 12	0.000 13	−0.000 11
100	300	0.000 08	−0.000 05	0.000 08	−0.000 21
100	400	−0.000 66	0.000 64	−0.000 66	−0.000 45
300	400	0.000 02	−0.000 02	0.000 02	−0.000 02
300	500	0.000 07	−0.000 07	0.000 07	−0.000 06
300	600	0.000 12	−0.000 12	0.000 12	−0.000 10
500	600	0.000 01	−0.000 01	0.000 01	−0.000 01
500	700	0.000 03	−0.000 03	0.000 03	−0.000 03
500	800	0.000 06	−0.000 06	0.000 06	−0.000 05

TABLE VI. (a) Results of $n=2$ analytical approximation for lens matrix elements, and (b) fractional error in (a) relative to numerical results in Table II.

V_1 (kV)	V_2 (kV)	a_{11}	a_{12}/b	$a_{21}b$	a_{22}
			(a)		
100	200	0.819 01	0.006 21	−0.029 96	0.827 07
100	300	0.715 00	0.016 20	−0.064 37	0.742 44
100	400	0.640 72	0.027 48	−0.090 71	0.689 29
300	400	0.913 48	0.001 03	−0.005 57	0.914 10
300	500	0.848 04	0.003 16	−0.016 19	0.851 21
300	800	0.795 21	0.005 75	−0.027 77	0.802 57
500	600	0.940 28	0.000 42	−0.002 29	0.940 44
500	700	0.890 91	0.001 38	−0.007 38	0.891 84
500	800	0.848 81	0.002 64	−0.013 71	0.851 22
			(b)		
100	200	0.000 00	−0.008 32	0.000 13	0.000 00
100	300	−0.000 01	0.024 27	0.000 08	−0.000 04
100	400	0.000 01	0.073 66	−0.000 66	−0.000 40
300	400	0.000 00	−0.018 17	0.000 02	0.000 00
300	500	0.000 00	−0.014 56	0.000 07	0.000 00
300	600	0.000 00	−0.008 46	0.000 12	0.000 00
500	600	0.000 00	−0.018 94	0.000 01	0.000 00
500	700	0.000 00	−0.017 67	0.000 03	0.000 00
500	800	0.000 00	−0.015 54	0.000 06	0.000 00

ditional expressions used in the definitions of $\tilde{S}_3$ and $\tilde{S}_4$ are

$$g_0(\gamma_1,\gamma_2)=\bar{G}(0)=\frac{\Delta\gamma^2}{4}\frac{[(\gamma_1+\gamma_2)^2+8]}{[(\gamma_1+\gamma_2)^2-4]^2}\ , \tag{40}$$

$$g_1(\gamma_1,\gamma_2)=\left.\frac{d\bar{G}}{d\xi}\right|_{\xi=1}=-\frac{\Delta\gamma^2}{8}\frac{(\gamma_2^2+2)}{(\gamma_2^2-1)^2}\ , \tag{41}$$

$$f_0(\gamma_1,\gamma_2)=\bar{F}(0)=-\frac{1}{4}-\frac{3}{4}\frac{[\gamma_2(\gamma_1+\gamma_2)-2]}{[(\gamma_1+\gamma_2)^2-4]}+\frac{[(\gamma_1+2)(\gamma_2+2)-9]}{8\Delta\gamma}\ln\left[\frac{\gamma_1+\gamma_2-2}{2(\gamma_1-1)}\right]-\frac{[(\gamma_1-2)(\gamma_2-2)-9]}{8\Delta\gamma}\ln\left[\frac{\gamma_1+\gamma_2+2}{2(\gamma_1+1)}\right]\ . \tag{42}$$

We note also the relation

$$f_0(\gamma_1,\gamma_2)+f_0(\gamma_2,\gamma_1)=S_1(\gamma_1,\gamma_2)\ . \tag{43}$$

In terms of Eqs. (40)–(43),

$$\tilde{S}_3(\gamma_1,\gamma_2)=-f_0^2(\gamma_1,\gamma_2)+\tfrac{1}{2}S_1^2(\gamma_1,\gamma_2)-\tfrac{1}{24}S_1(\gamma_1,\gamma_2)g_1(\gamma_1,\gamma_2)\ , \tag{44}$$

$$\tilde{S}_4(\gamma_1,\gamma_2)=\tfrac{1}{6}S_1^2(\gamma_1,\gamma_2)+[\tfrac{8}{3}\ln 2-2]f_0(\gamma_1,\gamma_2)g_0(\gamma_1,\gamma_2)+\tfrac{1}{6}(\tfrac{1}{2}-\ln 2)S_1(\gamma_1,\gamma_2)g_1(\gamma_1,\gamma_2)\ . \tag{45}$$

Results of calculations using the $n=2$ formulas, Eqs. (36)–(45), are shown in Table V and VI. Excellent agreement with results from numerical integration of Eq. (3) is apparent over the entire range of values of V_1 and V_2 considered. The $n=2$ expressions have been implemented in a beam transport code used to model the Dual Axis Radiographic Hydro-Test (DARHT) linear accelerators at Los Alamos National Laboratory [4].

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